MAGIC SQUARES.

MAGIC squares are of themselves only mathematical curios, but they involve principles whose unfolding should lead the thoughtful mind to a higher conception of the wonderful laws of symphony and order which govern the science of numbers.

The earliest record of a magic square is found in Chinese literature dated about A. D. 1125,* but since then this interesting subject has been more or less studied and developed by mathematicians of all nations.

It is the writer's purpose to present some general and comprehensive methods for constructing magic squares which he believes to be original, and also to briefly review what is commonly known concerning their construction.

THE GENERAL QUALITIES AND CHARACTERISTICS OF MAGIC SQUARES.

A magic square consists of a series of numbers arranged in quadratic form so that the sum of each vertical, horizontal and corner diagonal column is the same amount. These squares can be made with either an odd or an even number of cells, but as odd squares are constructed by methods which differ from those that govern the formation of even squares, the two classes will be considered under separate headings.

ODD MAGIC SQUARES.

In these squares it is not only requisite that the sum of all columns shall be the same amount, but also that the sum of any

* See p. 19 of *Chinese Philosophy* by Dr. Paul Carus.

two numbers that are geometrically equidistant from the center number shall equal twice that number. Unless these conditions are fulfilled, the square cannot be considered perfect.

The square of 3×3 shown in Fig. 1 covers the smallest aggregation of numbers that is capable of magic square arrangement, and it is also the only possible arrangement of nine different numbers, relatively to each other, which fulfills the required conditions. It will be seen that the sum of each of the three vertical, the three horizontal, and the two corner diagonal columns in this square is 15, making in all eight columns having that total: also that the sum of any two opposite numbers is 10, which is twice the center number. It is therefore a perfect square of 3×3.

The next largest odd magic square is that of 5×5, and there are a great many different arrangements of twenty-five numbers,

8	1	6
3	5	7
4	9	2

Totals = 15.

Fig 1.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Totals = 65.

Fig. 2.

which will show perfect results, each arrangement being the production of a different constructive method. Fig. 2. illustrates what is probably the oldest and best known arrangement of this square.

The sum of each of the five horizontal, the five vertical, and the two corner diagonal columns is 65, and the sum of any two numbers which are geometrically equidistant from the center number is 26, or twice the center number.

In order to intelligently follow the rule used in the construction of this square it may be conceived that its upper and lower edges are bent around backwards, and united to form a horizontal cylinder with the numbers on the outside, the lower line of figures thus coming next in order to the upper line. It may also be conceived that the square is bent around backwards in a direction at right

angles to that which was last considered, so that it forms a vertical cylinder with the extreme right and left hand columns adjacent to each other.

An understanding of this simple conception will assist the student to follow the new methods of building odd magic squares that are to be described, all of these methods being based on a right or left hand diagonal formation.

Referring to Fig. 2, it will be seen that the square is started by writing unity in the center cell of the upper row, the consecutive numbers proceeding diagonally therefrom in a right hand direction. Using the conception of a horizontal cylinder, 2 will be located in the lower row, followed by 3 in the next upper cell to the right. Here the formation of the vertical cylinder being conceived the next upper square will be where 4 is written, then 5; further progress being here blocked by 1 which already occupies the next upper cell in diagonal order.

When a block thus occurs in the regular spacing (which will be at every fifth number in a 5×5 square) the next number must in this case be written in the cell vertically below the one last filled, so that 6 is written in the cell below 5, and the right hand diagonal order is then continued in cells occupied by 7 and 8. Here the horizontal cylinder is imagined, showing the location of 9, then the conception of the vertical cylinder will indicate the location of 10; further regular progression being here once more blocked by 6, so 11 is written under 10 and the diagonal order continued to 15. A mental picture of the combination of vertical and horizontal cylinders will here show that further diagonal progress is blocked by 11, so 16 is written under 15. The vertical cylinder will then indicate the cell in which 17 must be located, and the horizontal cylinder will show the next cell diagonally upwards to the right to be occupied by 18, and so on until the final number 25 is reached and the square completed.

Fig. 3 illustrates the development of a 7×7 square constructed according to the preceding method, and the student is advised to follow the sequence of the numbers to impress the rule on his memory. A variation of the last method is shown in Fig. 4, illustrating

another 7×7 square. In this example 1 is placed in the next cell horizontally to the right of the center cell, and the consecutive numbers proceed diagonally upward therefrom, as before, in a right hand direction until a block occurs. The next number is then written in the second cell horizontally to the right of the last cell filled (instead of the cell below as in previous examples) and the upward diagonal order is resumed until the next block occurs.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Fig. 3.

4	29	12	37	20	45	28
35	11	36	19	44	27	3
10	42	18	43	26	2	34
41	17	49	25	1	33	9
16	48	24	7	32	8	40
47	23	6	31	14	39	15
22	5	30	13	38	21	46

Totals = 175.

Fig. 4.

10	18	1	14	22
11	24	7	20	3
17	5	13	21	9
23	6	19	2	15
4	12	25	8	16

Totals = 65.

Fig. 5.

Then two cells to the right again, and regular diagonal order continued, and so on until all the cells are filled.

The preceding examples may be again varied by writing the numbers in left hand instead of right hand diagonal sequence, making use of the same spacing of numbers as before when blocks occur in the regular sequence of construction.

We now come to a series of very interesting methods for building odd magic squares which involve the use of the knight's move in chess, and it is worthy of note that the squares formed by these methods possess curious characteristics in addition to those

previously referred to. To chess-players the knight's move will require no comment, but for those who are not familiar with this game it may be explained as a move of two squares straight forward in any direction and one square to either right or left.

The magic square of 5×5 illustrated in Fig. 5 is started by placing 1 in the center cell of the upper row, and the knight's move employed in its construction will be two cells upward and one cell to the right.

Using the idea of the horizontal cylinder 2 must be written in the second line from the bottom, as shown, and then 3 in the second line from the top. Now conceiving a combination of the horizontal and vertical cylinders, the next move will locate 4 in the extreme lower left hand corner, and then 5 in the middle row. We now find that the next move is blocked by one, so 6 is written below 5, and the knight's moves are then continued, and so on until the last number, 25, is written in the middle cell of the lower line, and the square is thus completed.

In common with the odd magic squares which were previously described, it will be found that in this square the sum of each of the five horizontal, the five perpendicular, and the two corner diagonal columns is 65, also that the sum of any two numbers that are geometrically equidistant from the center is 26, or twice the number in the center cell, thus filling all the general qualifications of a perfect square.

In addition, however, to these characteristics it will be noted that each spiral row of figures around the horizontal and vertical cylinders traced either right handed or left handed also amounts to 65. In the vertical cylinder, there are five right hand, and five left hand spirals, two of which form the two corner diagonal columns across the square, leaving eight new combinations. The same number of combinations will also be found in the horizontal cylinder. Counting therefore five horizontal columns, five vertical columns, two corner diagonal columns, and sixteen right and left hand spiral columns, there will be found in all twenty-eight columns each of which will sum up to 65, whereas in either of the 5×5

squares previously considered there will be found only twelve columns that will amount to that number.

This method of construction is subject to a number of variations. For example, the knight's move may be upwards and to the left hand instead of to the right, or it may be made downwards and either to the right or left hand, and also in other directions. There are in fact eight different ways in which the knight's move may be started from the center cell in the upper line. Six of these moves are indicated by figure 2's in different cells of Fig. 6, and each of these moves if continued in its own direction, varied by regular breaks as before described, will produce a different but perfect square. The remaining two possible knight's moves, indicated by cyphers, will not produce perfect squares.

		1		
0				0
	2		2	
	2		2	
2				2

Fig. 6.

		19	2	15	23	
	12	25	8		4	
10	18	1	14	22	10	
11	24	7	20	3		
17	5	13	21	9	17	
23	6	19	2	15		
4	12	25	8	16		

Fig. 7.

It may here be desirable to explain another method for locating numbers in their proper cells which some may prefer to that which involves the conception of the double cylinder. This method consists in constructing parts of auxiliary squares around two or more sides of the main square, and temporarily writing the numbers in the cells of these auxiliary squares when their regular placing carries them outside the limits of the main square. The temporary location of these numbers in the cells of the auxiliary squares will then indicate into which cells of the main square they must be permanently transferred.

Fig. 7 shows a 5×5 main square with parts of three auxiliary

squares, and the main square will be built up in the same way as Fig. 5.

Starting with 1 in the center of the top line, the first knight's move of two cells upwards and one to the right takes 2 across the top margin of the main square into the second cell of the second line from the bottom in one of the auxiliary squares, so 2 must be transferred to the same relative position in the main square. Starting again from 2 in the main square, the next move places 3 within the main square, but 4 goes out of it into the lower left hand corner of an auxiliary square, from which it must be transferred to the same location in the main square, and so on throughout.

The method last described and also the conception of the double cylinders may be considered simply as aids to the beginner. With a little practice the student will be able to select the proper cells in the square as fast as the figures can be written therein.

Having thus explained certain specific and novel lines of construction, the general principles governing the development of all odd magic squares by these methods may now be formulated.

1. The center cell in the square must always contain the middle number of the series of numbers used, i. e., a number which is equal to one half the sum of the first and last numbers of the series.
2. No perfect magic square can therefore be started from its center cell, but it may be started from any cell other than the center one.
3. With certain specific exceptions which will be referred to later on, odd magic squares may be constructed by either right or left hand diagonal sequence, or by a number of so-called knight's moves, varied in all cases by periodical and well defined departures from normal spacing.
4. The directions and dimensions of these departures from normal spacing, or "break moves," as they may be termed, are governed by the relative spacing of cells occupied by the first and last numbers of the series, and may be determined as follows:

RULE: Place the first number of the series in any desired cell (excepting the center one) and the last number of the series in the cell which is geometrically opposite to the cell containing the first number. The relative spacing of these two cells must then be repeated whenever a block occurs in the regular progression.

EXAMPLES.

Using a blank square of 5 × 5, 1 may be written in the middle cell of the upper line. The geometrically opposite cell to this being the middle cell in the lower line, 25 must be written therein. 1 will therefore be located four cells above in the middle vertical column, or what is the same thing, and easier to follow, one cell below 25. When, therefore, a square of 5 × 5 is commenced with the first number in the middle cell of the upper line, the break move will *always* be one cell downwards, irrespective of the method of regular

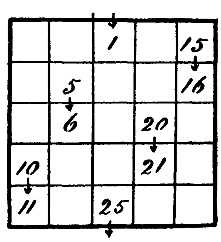

Fig. 8.

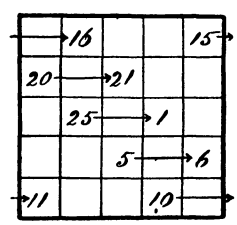

Fig. 9.

advance. Fig. 8 shows the break moves in a 5 × 5 square as above described using a right hand upward diagonal advance.

Again using a blank 5 × 5 square, 1 may be written in the cell immediately to the right of the center cell, bringing 25 into the cell to the left of the center cell. The break moves in this case will therefore be two cells to the right of the last cell occupied, irrespective of the method used for regular advance. Fig. 9 illustrates the break moves in the above case, when a right hand upward diagonal advance is used. The positions of these break moves in the square will naturally vary with the method of advance, but the spacing of the moves themselves will remain unchanged.

NOTE: The foregoing break moves were previously described in several specific examples (See Figs. 1, 2, 3, 4, and 5) and the student will now observe how they agree with the general rule.

Once more using a blank square of 5 × 5, 1 may be written in the upper left hand corner and 25 in the lower right hand corner. 1 will then occupy a position four cells removed from 25 in a left hand upward diagonal, or what is the same thing and easier to follow, the next cell in a right hand downward diagonal. This will therefore be the break move whenever a block occurs in the regular spacing. (See Fig. 10.)

As a final example we will write 1 in the second cell from the left in the upper line of a 5 × 5 square, which calls for the placing

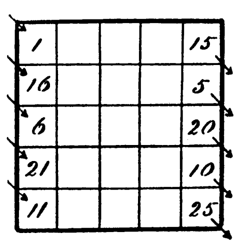

Fig. 10.

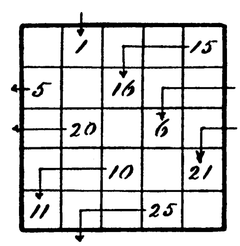

Fig. 11.

of 25 in the second square from the right in the lower line. The place relation between 25 and 1 may then be described by a knight's move of two cells to the left and one cell downwards, and this must be the break move whenever a block occurs in the regular spacing. (See Fig. 11.)

As before stated odd magic squares may be commenced in any cell excepting the center one, and perfect squares may be built up from such commencements by a great variety of regular moves, such as right hand diagonal sequence, upwards or downwards, left hand diagonal sequence upwards or downwards, or a number of knight's moves in various directions. There are four possible moves from each cell in diagonal sequence, and eight possible moves from each cell by the knight's move. The greater number of these moves

will produce perfect magic squares, but there will be found certain exceptions which can be shown most readily by diagrams.

Fig. 12 is a 5 × 5 square in which the pointed arrow heads indicate the directions of diagonal sequence by which perfect squares may be constructed, while the blunt arrow heads show the directions

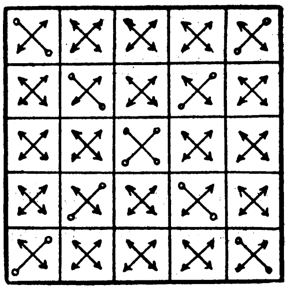

Fig. 12.

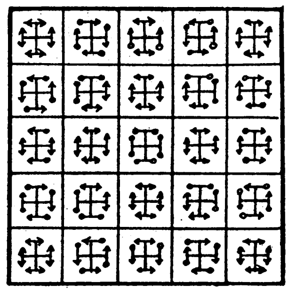

Fig. 13.

of diagonal sequence which will lead to mperfect results. Fig. 13 illustrates the various *normal* knight's moves which may be started from each cell and also indicates with pointed and blunt arrow heads the moves which will lead to perfect or imperfect results.

EXAMPLES OF 5 × 5 MAGIC SQUARES.

Figs. 14, 15, and 16 show three 5 × 5 squares, each having 1 in the upper left hand corner cell and 25 in the lower right hand corner cell, and being constructed respectively with a right hand

1	17	8	24	15
16	7	23	14	5
6	22	13	4	20
21	12	3	19	10
11	2	18	9	25

Fig. 14.

1	21	16	11	6
12	7	2	22	17
23	18	13	8	3
9	4	24	19	14
20	15	10	5	25

Fig. 15.

1	11	21	6	16
22	7	17	2	12
18	3	13	23	8
14	24	9	19	4
10	20	5	15	25

Fig. 16.

upward diagonal sequence and right and left hand horizontal knight's moves, the break move being necessarily the same in each example. (See Fig. 10.)

Figs. 17, 18, 19, and 20 show four 5×5 squares, each having 1 in the second cell from the left in the upper line and 25 in the second cell from the right in the lower line, and being built up respectively with right and left hand upward diagonal sequence and upward right and downward left hand knight's moves, and with similar break moves in each example. (See Fig. 11.)

8	1	24	17	15
5	23	16	14	7
22	20	13	6	4
19	12	10	3	21
11	9	2	25	18

Fig. 17.

15	1	17	8	24
23	14	5	16	7
6	22	13	4	20
19	10	21	12	3
2	18	9	25	11

Fig. 18.

22	1	10	14	18
11	20	24	3	7
5	9	13	17	21
19	23	2	6	15
8	12	16	25	4

Fig. 19.

23	1	9	12	20
15	18	21	4	7
2	10	13	16	24
19	22	5	8	11
6	14	17	25	3

Fig 20.

18	10	22	14	1
11	3	20	7	24
9	21	13	5	17
2	19	6	23	15
25	12	4	16	8

Fig. 21.

9	12	20	23	1
18	21	4	7	15
2	10	13	16	24
11	19	22	5	8
25	3	6	14	17

Fig. 22.

12	23	9	20	1
4	15	21	7	18
16	2	13	24	10
8	19	5	11	22
25	6	17	3	14

Fig. 23.

Figs. 21, 22, and 23 illustrate three 5×5 squares, each having 1 in the upper right hand corner and 25 in the lower left hand

corner, and being built up respectively with upward and downward right hand normal knight's moves, and a downward right hand elongated knight's move.

For the sake of simplicity these examples have been shown in 5×5 squares, but the rules will naturally apply to all sizes of odd magic squares by using the appropriate numbers. The explanations have also been given at some length because they cover general and comprehensive methods, a good understanding of which will make the student a master of the entire subject of odd magic squares.

It is clear that no special significance can be attached to the so-called knight's move, *per se,* as applied to the construction of magic squares, it being only one of many methods of regular spacing, all of which will produce equivalent results. For example, the 3×3 square shown in Fig. 1 may be said to be built up by a succession of abbreviated knight's moves of one cell to the right and one cell upwards. Squares illustrated in Figs. 2, 3, and 4 are also constructed by this abbreviated knight's move, but the square illustrated in Fig. 5 is built up by the normal knight's move.

80	58	45	23	1	69	47	34	12
9	68	46	33	11	79	57	44	22
10	78	56	43	21	8	67	54	32
20	7	66	53	31	18	77	55	42
30	17	76	63	41	19	6	65	52
40	27	5	64	51	29	16	75	62
50	28	15	74	61	39	26	4	72
60	38	25	3	71	49	36	14	73
70	48	35	13	81	59	37	24	2

Totals = 369.

Fig. 24.

It is equally easy to construct squares by means of an elongated knight's move, say, four cells to the right and one cell upwards as shown in Fig. 24, or by a move consisting of two cells to the right and two cells downwards, as shown in Fig. 25, the latter being

equivalent to a right hand downward diagonal sequence wherein alternate cells are consecutively filled.

There are in fact almost innumerable combinations of moves by which perfect odd magic squares may be constructed.

39	34	20	15	1	77	72	58	53
49	44	30	25	11	6	73	68	63
59	54	40	35	21	16	2	78	64
69	55	50	45	31	26	12	7	74
79	65	60	46	41	36	22	17	3
8	75	70	56	51	37	32	27	13
18	4	80	66	61	47	42	28	23
19	14	9	76	71	57	52	38	33
29	24	10	5	81	67	62	48	43

Totals = 369.

Fig. 25.

The foregoing methods for building odd magic squares by a continuous process, involving the regular spacing of consecutive numbers varied by different well defined break moves is believed to be new and original with the writer, but other methods of construction have been known for many years.

One of the most interesting of these older methods involves the use of two or more primary squares, the sums of numbers in similarly located cells of which constitute the correct numbers for transfer into the corresponding cells of the magic square that is to be constructed therefrom.

This method has been ascribed primarily to De la Hire but has been more recently improved by Prof. Scheffler.

It may be simply illustrated by the construction of a few 5×5 squares as examples. Figs. 26 and 27 show two simple primary squares in which the numbers 1 to 5 are so arranged that like numbers occur once and only once in similarly placed cells in the two squares; also that pairs of unlike numbers are not repeated in the same order in any similarly placed cells. Thus, 5 occupies the extreme right hand cell in the lower line of each square, but this com-

bination does not occur in any of the other cells. So also in Fig. 27 3 occupies the extreme right hand cell in the upper line, and in Fig. 26 this cell contains 5. No other cell, however, in Fig. 27 that contains 3 corresponds in position with a cell in 26 that contains 5. Leaving the numbers in Fig. 26 unaltered, the numbers in Fig. 27 must now be changed to their respective key numbers, thus producing the key square shown in Fig. 28. By adding the cell numbers of the primary square Fig. 26 to the corresponding cell numbers

Prime numbers,. . . .1, 2, 3, 4, 5.
Key numbers,. . . . .0, 5. 10, 15, 20.

1	2	3	4	5
1	2	3	4	5
1	2	3	4	5
1	2	3	4	5
1	2	3	4	5

Fig. 26.

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Fig. 27.

0	15	5	20	10
15	5	20	10	0
5	20	10	0	15
20	10	0	15	5
10	0	15	5	20

Fig. 28.

1	17	8	24	15
16	7	23	14	5
6	22	13	4	20
21	12	3	19	10
11	2	18	9	25

Fig. 29.

of the key square Fig. 28, the magic square shown in Fig. 29 is formed, which is also identical with the one previously given in Fig. 14.

The simple and direct formation of Fig. 14 may be thus compared with the De la Hire method for arriving at the same result.

It is evident that the key square shown in Fig. 28 may be dispensed with by mentally substituting the key numbers for the prime

numbers given in Fig. 27 when performing the addition, and by so doing only two primary squares are required to construct the magic square. The arrangement of the numbers 1 to 5 in the two primary squares is obviously open to an immense number of variations, each of which will result in the formation of a different but perfect magic square. Any of these squares, however, may be more readily constructed by the direct methods previously explained.

A few of these variations are given as examples, the key numbers remaining unchanged. The key square Fig. 32 is formed from the primary square Fig. 31, and if the numbers in Fig. 32 are added to those in the primary square Fig. 30, the magic square given in Fig. 33 will be produced. This square will be found identical with that shown in Fig. 15.

1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5

Fig. 30.

1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3
4	3	2	1	5

Fig. 31.

0	20	15	10	5
10	5	0	20	15
20	15	10	5	0
5	0	20	15	10
15	10	5	0	20

Fig. 32.

1	21	16	11	6
12	7	2	22	17
23	18	13	8	3
9	4	24	19	14
20	15	10	5	25

Fig. 33.

Fig. 30 cannot be used as a key square, but if two primary squares are constructed in which every horizontal and perpendicular column contains the numbers 1 to 5 placed according to rules previously given, and having a different arrangement of numbers in each primary square, then either of these squares may be made

the key square, and two different magic squares may be constructed therefrom, as shown in the next examples.

The magic square shown in Fig. 37 is made by the addition of numbers in the primary square Fig. 34 to the numbers occupying similar cells in the key square Fig. 36, the latter being derived from the primary square Fig. 35. If the key square shown in Fig. 38 is now constructed from the primary square Fig. 34 and the key numbers therein added to the prime numbers in Fig. 35, the magic square shown in Fig. 39 is obtained. This square has not been given before in this treatise, but it may be directly produced by

3	1	4	2	5
5	3	1	4	2
2	5	3	1	4
4	2	5	3	1
1	4	2	5	3

Fig. 34.

2	1	5	4	3
1	5	4	3	2
5	4	3	2	1
4	3	2	1	5
3	2	1	5	4

Fig. 35.

5	0	20	15	10
0	20	15	10	5
20	15	10	5	0
15	10	5	0	20
10	5	0	20	15

Fig. 36.

8	1	24	17	15
5	23	16	14	7
22	20	13	6	4
19	12	10	3	21
11	9	2	25	18

Fig. 37.

10	0	15	5	20
20	10	0	15	5
5	20	10	0	15
15	5	20	10	0
0	15	5	20	10

Fig. 38.

12	1	20	9	23
21	15	4	18	7
10	24	13	2	16
19	8	22	11	5
3	17	6	25	14

Fig. 39.

an elongated knight's move consisting of two cells to the right and two downwards, using the normal knight's move of two cells to the left and one cell downwards as a break move at every block in the regular spacing.

It will be observed in all the preceding examples that the number 3 invariably occupies the center cell in every primary square, thus bringing 10 in the center of all key squares, and 13 in the center of magic squares, no other number being admissible in the

center cell of a 5×5 magic square. A careful study of these examples should suffice to make the student familiar with the De la Hire system for building odd magic squares, and this knowledge is desirable in order that he may properly appreciate the more direct methods which have been described.

Before concluding this branch of the subject, mention may be made of another method for constructing odd magic squares which is said to have been originated by Bachet de Mezeriac. The application of this method to a 5×5 square will suffice for an example.

The numbers 1 to 25 are written consecutively in diagonal columns, as shown in Fig. 40, and those numbers which come out-

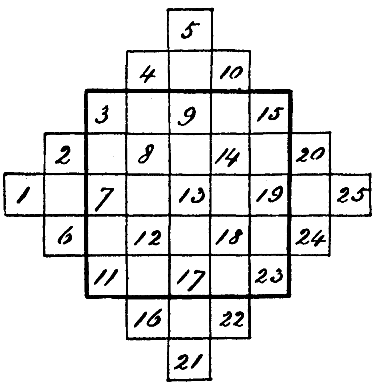

Fig. 40.

3	16	9	22	15
20	8	21	14	2
7	25	13	1	19
24	12	5	18	6
11	4	17	10	23

Fig. 41.

side the center square are transferred to the empty cells on the opposite sides of the latter without changing their order. The result will be the magic square of 5×5 shown in Fig. 41. It will be seen that the arrangement of numbers in this magic square is similar to that in the 7×7 square shown in Fig. 4, which was built by writing the numbers 1 to 49 consecutively according to rule. The 5×5 square shown in Fig. 41 may also be written out directly by the same rule without any preliminary or additional work.

EVEN MAGIC SQUARES.

In perfect squares of this class it is necessary that the sum of each column shall be the same amount, and also that the sum of

any two numbers that are geometrically equidistant from the center of the square shall equal the sum of the first and last numbers of the series.

The numbers in the two corner diagonal columns in even magic squares may be determined by writing the numbers of the series in arithmetical order in horizontal columns, beginning with the first number in the left hand cell of the upper line and writing line after line as in a book, ending with the last number in the right hand cell of the lower line. The numbers then found in the two diagonal columns will be in magic square order, but the position of the other numbers must generally be changed.

The smallest even magic square that can be built is that of 4×4, and one of its forms is shown in Fig. 42. It will be seen

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Fig. 42.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 43.

that the sum of each of the four horizontal, the four vertical, and the two corner diagonal columns in this square is 34, making in all ten columns having that total; also that the sum of any two geometrically opposite numbers is 17, which is the sum of the first and last numbers of the series. It is therefore a pertect square of 4×4.

The first step in the construction of this square is shown in Fig. 43, in which only the two corner diagonal columns, which are written in heavy figures, have the correct summation. The numbers in these two columns must therefore be left as they are, but the location of all the other numbers, which are written in light figures, must be changed. A simple method for effecting this change consists in substituting for each number the complement between it and 17. Thus, the complement between 2 and 17 is 15, so 15 must be written in the place of 2, and so on throughout. All of the light figure

numbers being thus changed, the result will be the perfect magic square shown in Fig 42.

The same relative arrangement of figures may be attained by leaving the light figure numbers in their original positions as shown in Fig. 43, and changing the heavy figure numbers in the two corner diagonal columns to their respective complements with 17. It will be seen that this is only a reversal of the order of the figures

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Fig. 44.

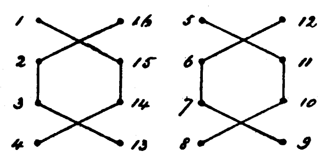

Fig. 45..

in the two corner diagonal columns, and the resulting magic square which is shown in Fig. 44 is simply an inversion of Fig. 42.

Fig. 45 is a geometrical diagram of the numbers in Fig. 42, and it indicates a regular law in their arrangement, which also holds good in many larger even squares, as will be seen later on.

There are many other arrangements of sixteen numbers which will fulfil the required conditions but the examples given will suffice to illustrate the principles of this square.

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Fig. 46.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Fig. 47.

The next even magic square is that of 6×6, and one of its many variations is shown in Fig. 46. An analysis of this square with the aid of geometrical diagrams will point the way not only

to its own reconstruction but also to an easy method for building 6×6 squares in general.

Fig. 47 shows a 6×6 square in which all the numbers from 1 to 36 are written in arithmetical sequence, and the twelve numbers in the two corner diagonal columns will be found in magic square order, all other numbers requiring rearrangement. Leaving there-

1	35	34	33	32	6
30	8	28	27	11	25
24	23	15	16	20	19
18	17	21	22	14	13
12	26	10	9	29	7
31	5	4	3	2	36

Fig. 48.

fore the numbers in the diagonal columns unchanged, the next step will be to write in the places of the other numbers their complements with 37, making the square shown in Fig. 48. In this square twenty-four numbers (written in heavy figures) out of the total of

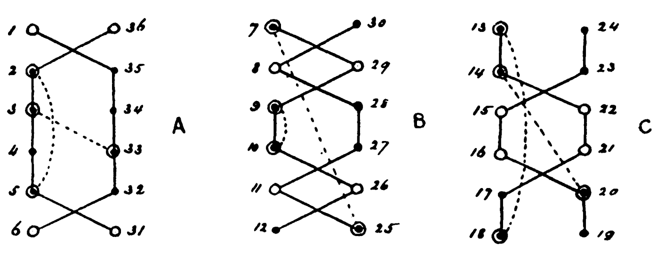

Fig. 49.

thirty-six numbers, will be found in magic square order, twelve numbers (written in light figures) being still incorrectly located. Finally, the respective positions of these twelve numbers being reversed in pairs, the magic square given in Fig. 46 will be produced.

Fig. 50 shows the geometrical diagrams of this square, A being a diagram of the first and sixth lines, B of the second and fifth lines, and C of the third and fourth lines. The striking ir-

regularity of these diagrams points to the imperfection of the square which they represent, in which, although the sum of each of the two corner diagonal, the six horizontal, and the six perpendicular columns is 111, yet only in the two diagonal columns does the sum of any two numbers which occupy geometrically opposite cells, amount to 37, or the sum of the first and last numbers of the series. Owing to their pronounced irregularities, these diagrams convey but little meaning, and in order to analyze their value for further constructive work it will be necessary to go a step backwards and make diagrams of the intermediate square Fig. 48. These diagrams are shown in Fig. 49, and the twelve numbers therein which must be transposed (as already referred to) are marked by small circles around dots, each pair of numbers to be transposed in position

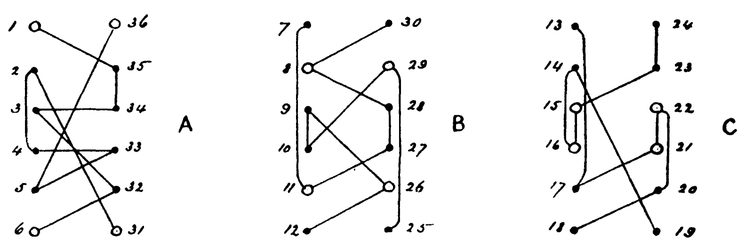

Fig. 50.

being connected by a dotted line. The numbers in the two corner diagonal columns which were permanently located from the beginning are marked with small circles.

We have here correct geometrical figures with definite and well defined irregularities. The series of geometrical figures shown in A, B, and C remain unchanged in shape for all variations of 6×6 squares, but by modifying the irregularities we may readily obtain the data for building a large number of different 6×6 squares, all showing, however, the same general characteristics as Fig. 46.

A series of these diagrams, with some modifications of their irregularities, is given in Fig. 51, and in order to build a variety of 6×6 magic squares therefrom it is only necessary to select three diagrams in the order A, B, and C, *which have each a different form*

of irregularity, and after numbering them in arithmetical sequence from 1 to 36, as shown in Fig. 49, copy the numbers in *diagrammatic order* into the cells of a 6×6 square.

It must be remembered that the cells in the corner diagonal columns of even magic squares may be correctly filled by writing

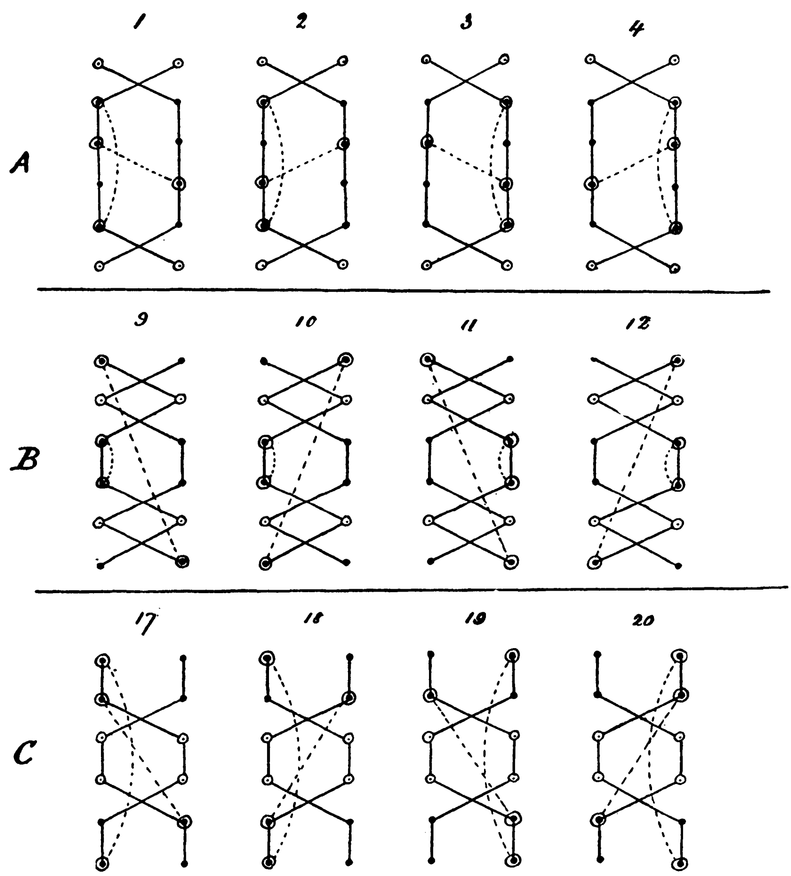

Fig. 51 (First Part).

the numbers in arithmetical order according to the rule previously given, so in beginning any new even square it will be found helpful to first write the numbers in these columns, and they will then serve as guides in the further development of the square.

Taking for example the 6×6 magic square shown in Fig. 46,

it will be seen from Fig. 49 that it is constructed from the diagrams marked 1—9 and 17 in Fig. 51. Comparing the first line of Fig. 46 with diagram A, Fig. 49, the sequence of numbers is 1,—35,—34 in unbroken order; then the diagram shows that 33 and 3 must be transposed, so 3 is written next (instead of 33) then 32 and 6 in

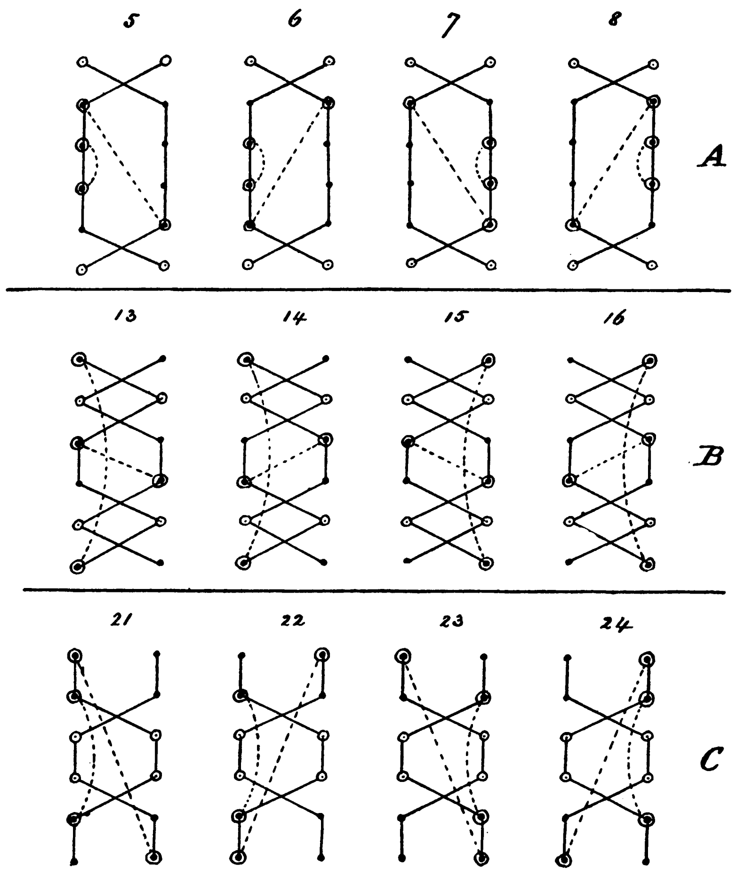

Fig. 51 (Second Part).

unbroken order. In the last line of this square (still using diagram A) 31 comes first, then, seeing that 5 and 2 must be transposed, 2 is written instead of 5; then 4; then as 3 and 33 must be transposed, 33 is written instead of 3, 5 instead of 2, and the line is finished with 36. Diagram B gives the development of the second

and fifth lines of the square in the same manner, and diagram C the development of the third and fourth lines, thus completing the square.

The annexed table shows 128 changes which may be rung on the twenty-four diagrams shown in Figure 51, each combination giving a different 6×6 square, and many others might be added to the list.

TABLE SHOWING 128 CHANGES WHICH MAY BE RUNG ON THE TWENTY-FOUR DIAGRAMS IN FIG. 51.

A	B	C		
1, 2, 3 or 4	9	17, 18, 19 or 20	= 16	changes
" " " "	10	" " " "	= 16	"
" " " "	11	" " " "	= 16	"
" " " "	12	" " " "	= 16	"
5, 6, 7 or 8	13	21, 22, 23 or 24	= 16	"
" " " "	14	" " " "	= 16	"
" " " "	15	" " " "	= 16	"
" " " "	16	" " " "	= 16	"
		Total changes	= 128	"

EXAMPLES.

1	35	4	33	32	6
12	8	28	27	11	25
24	17	15	16	20	19
13	23	21	22	14	18
30	26	9	10	29	7
31	2	34	3	5	36

Square derived from diagrams 2, 10, and 18.

1	5	33	34	32	6
30	8	28	9	11	25
18	23	15	16	20	19
24	14	21	22	17	13
7	26	10	27	29	12
31	35	4	3	2	36

Square derived from diagrams 8, 13, and 22.

The next size of even magic square is that of 8×8, and instead of presenting one of these squares ready made and analyzing it, we will now use the information which has been offered by previous examples in the construction of a new square of this size.

Referring to Fig. 45, the regular geometrical diagrams of the 4×4 square naturally suggest that an expansion of the same may be utilized to construct an 8×8 square. This expanded diagram

is accordingly shown in Fig. 52, and in Fig. 53 we have the magic square that is produced by copying the numbers in diagrammatic order.

As might be anticipated, this square is perfect in all its characteristics, and the ease with which it has been constructed points to the simplicity of the method employed.

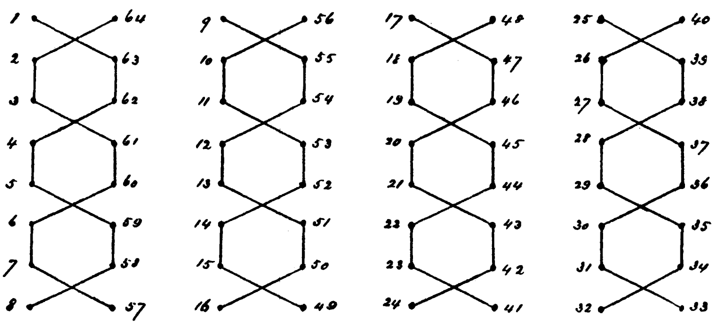

Fig. 52.

The magic square shown in Fig. 53 is, however, only one of a multitude of 8 × 8 squares, all of which have the same general

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Totals = 260.

Fig. 53.

characteristics and may be constructed with equal facility from various regular diagrams that can be readily derived from transpositions of Fig. 52. Five of these variations are illustrated in Fig. 54, which also show the transpositions by which they are formed from the original diagrams. To construct a perfect magic square

from either of these variations it is only necessary to make four copies of the one selected, annex the numbers 1 to 64 in arithmetical

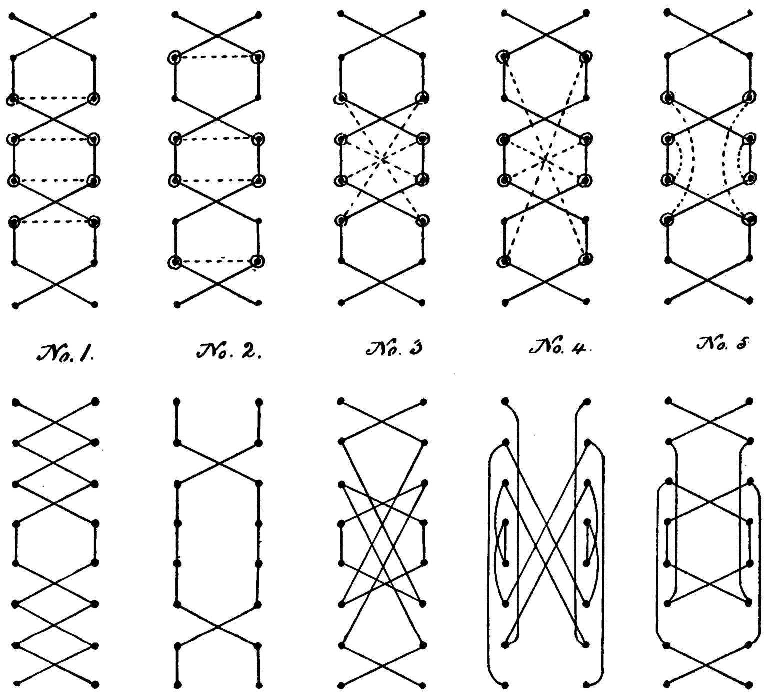

Fig. 54.

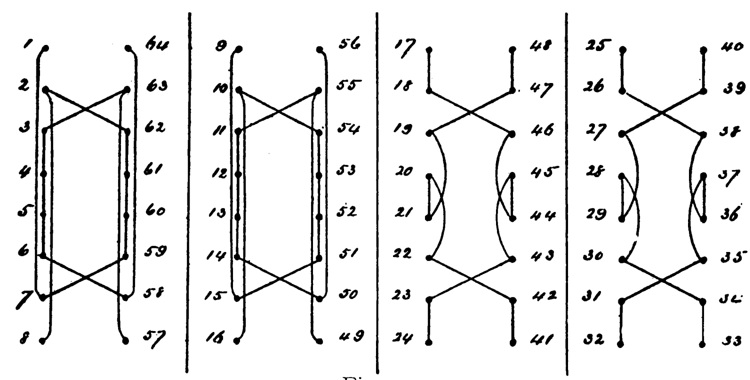

Fig. 55.

order as before explained, and then copy the numbers in diagrammatic sequence into the cells of an 8×8 square.

It will be noted in the construction of the 4×4 and 8×8 squares that only one form of diagram has been hitherto used for each square, whereas three different forms were required for the 6×6 square. It is possible, however, to use either two, three, or

1	7	59	60	61	62	2	8
16	10	54	53	52	51	15	9
48	47	19	21	20	22	42	41
33	34	30	28	29	27	39	40
25	26	38	36	37	35	31	32
24	23	43	45	44	46	18	17
56	50	14	13	12	11	55	49
57	63	3	4	5	6	58	64

Totals = 260.

Fig. 56.

four different diagrams in the construction of an 8×8 square, as shown in the annexed examples. Fig. 55 illustrates two different forms from which the magic square Fig. 56 is constructed. Fig. 57

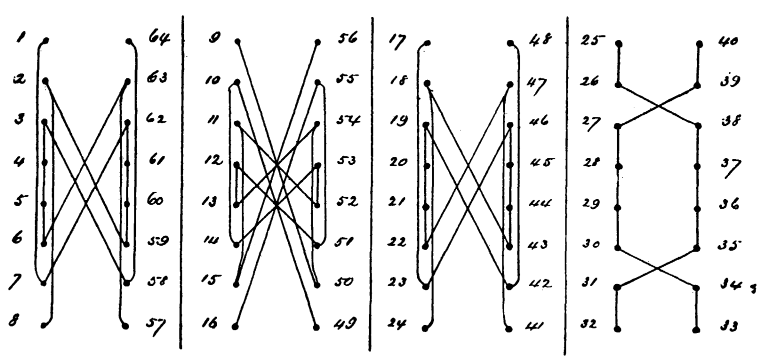

Fig. 57

shows three different forms which are used in connection with the square in Fig. 58, and in a similar manner Figs. 59 and 60 show four different diagrams and the square derived therefrom. The foregoing examples are sufficient to illustrate the immense number

1	7	62	61	60	59	2	8
49	10	14	53	52	11	15	56
48	42	19	20	21	22	47	41
40	39	27	28	29	30	34	33
32	31	35	36	37	38	26	25
24	18	43	44	45	46	23	17
9	50	54	13	12	51	55	16
57	63	6	5	4	3	58	64

Totals = 260.

Fig. 58.

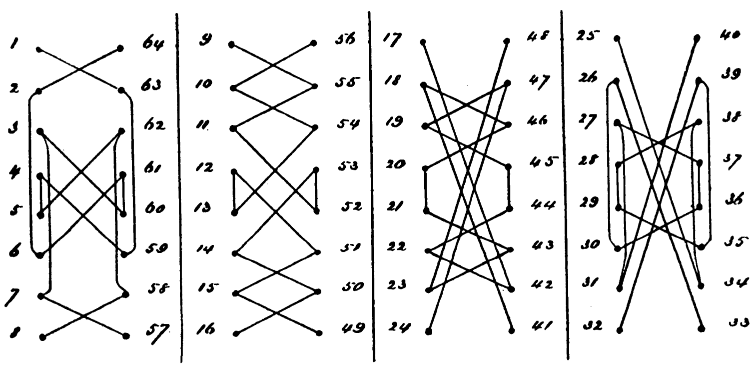

Fig. 59.

1	63	59	4	5	62	58	8
56	10	54	13	12	51	15	49
24	47	19	45	44	22	42	17
25	34	38	28	29	35	39	32
33	26	30	36	37	27	31	40
48	23	43	21	20	46	18	41
16	50	14	53	52	11	55	9
57	7	3	60	61	6	2	64

Totals = 260.

Fig. 60.

of different 8×8 magic squares that may be constructed by the aid of various diagrams.

We now come to the magic square of 10×10, and employing the comparative method of the last examples, it will be easy to expand the three diagrams of the 6×6 square (Fig. 49) into five

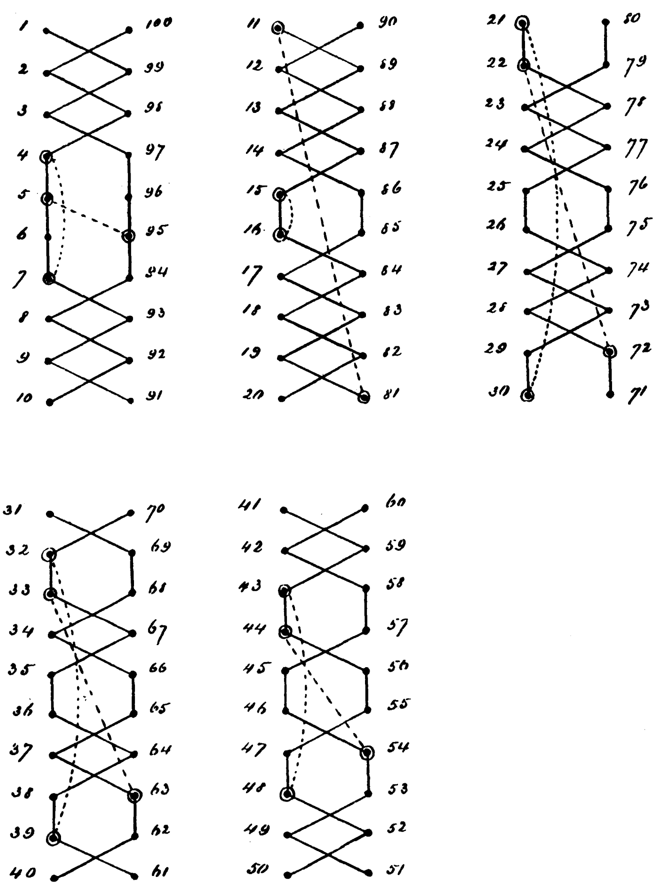

Fig. 61.

diagrams that are required for the construction of a series of 10×10 squares. These five diagrams are shown in Fig. 61, and in Fig. 62 we have the magic square which is made by copying the numbers from 1 to 100 in diagrammatic order into the cells of a 10×10 square.

It will be unnecessary to proceed further with the construction of other 10×10 squares, for the student will recognize the striking

1	99	3	97	96	5	94	8	92	10
90	12	88	14	86	85	17	83	19	11
80	79	23	77	25	26	74	28	22	71
31	69	68	34	66	65	37	33	62	40
60	42	58	57	45	46	44	53	49	51
50	52	43	47	55	56	54	48	59	41
61	32	38	64	36	35	67	63	39	70
21	29	73	27	75	76	24	78	72	30
20	82	18	84	15	16	87	13	89	81
91	9	93	4	6	95	7	98	2	100

Totals = 505.

Fig. 62.

resemblance between the diagrams of the 6×6 and the 10×10 squares, especially in connection with their respective irregularities,

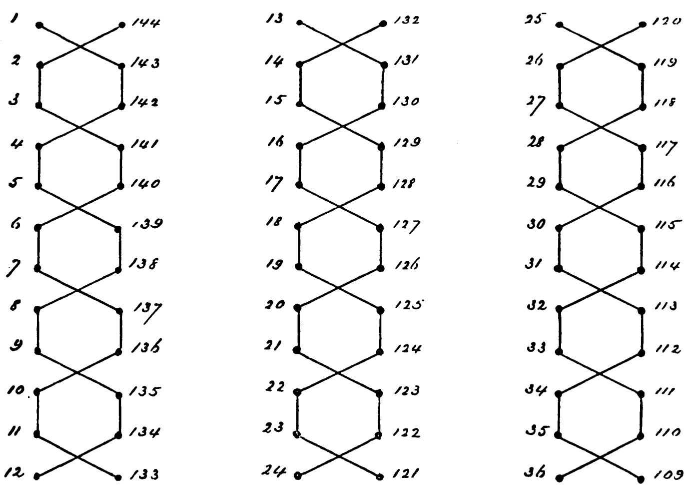

Fig. 63 (First part).

which point to the apparent impossibility of building perfect 10×10 squares.

It will also be seen that the same methods which were used for

varying the 6×6 diagrams, are equally applicable to the 10×10 diagrams, so that an almost infinite variety of changes may be rung on them, from which a corresponding number of 10×10 squares

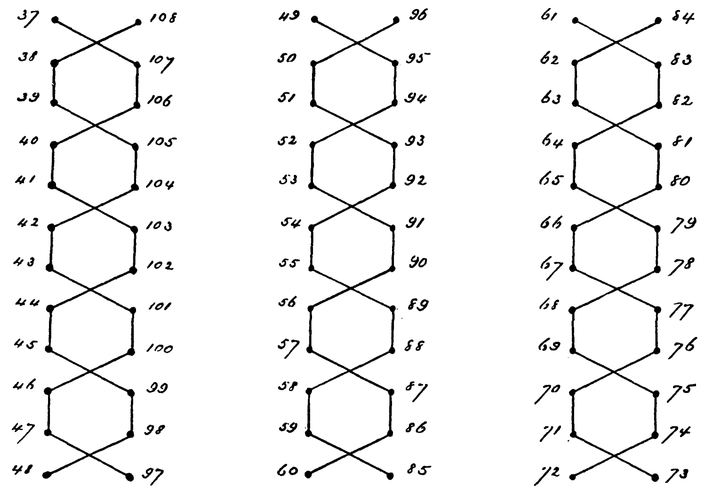

Fig. 63 (Second part).

1	143	142	4	5	139	138	8	9	135	134	12
132	14	15	129	128	18	19	125	124	22	23	121
120	26	27	117	116	30	31	113	112	34	35	109
37	107	106	40	41	103	102	44	45	99	98	48
49	95	94	52	53	91	90	56	57	87	86	60
84	62	63	81	80	66	67	77	76	70	71	73
72	74	75	69	68	78	79	65	64	82	83	61
85	59	58	88	89	55	54	92	93	51	50	96
97	47	46	100	101	43	42	104	105	39	38	108
36	110	111	33	32	114	115	29	28	118	119	25
24	122	123	21	20	126	127	17	16	130	131	13
133	11	10	136	137	7	6	140	141	3	2	144

Totals = 870

Fig. 64.

may be derived, each of which will be different but will resemble the series of 6×6 squares in their curious and characteristic imperfections.

We have thus far studied the construction of all even magic squares up to and including that of 10 × 10, and it is worthy of remark that when one half the number of cells in one side of an even magic square is an even number the square can be made perfect, but when it is an uneven number it is apparently impossible to build the square with perfect characteristics.

Even magic squares may therefore be divided into two classes —perfect and imperfect—the 4 × 4 and the 8 × 8 squares belong-

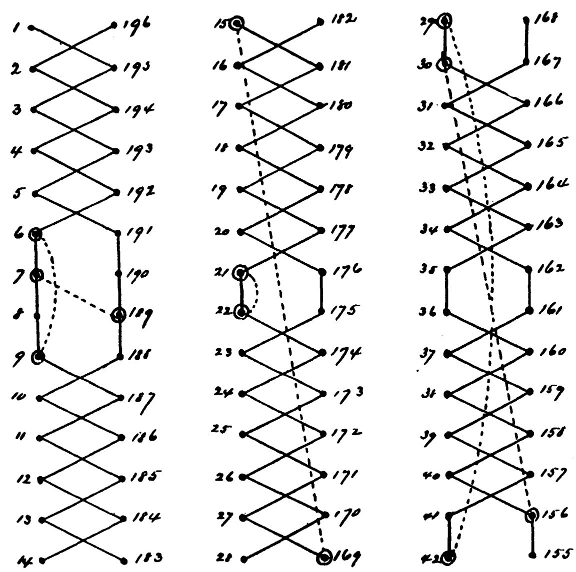

Fig. 65 (First Part).

ing to the first class, and the 6 × 6 and 10 × 10 belonging to the second class.

Fig. 63 shows a series of diagrams from which the 12 × 12 square in Fig. 64 is derived. The geometrical design of these diagrams is the same as that shown in Fig. 52 for the 8 × 8 square, and it is manifest that all the variations that were made in the 8 × 8 diagrams are also possible in the 12 × 12 diagrams, besides an immense number of additional changes which are allowed by the increased size of the square.

In Fig. 65 we have a series of diagrams illustrating the development of the 14 × 14 magic square shown in Fig. 66. These diagrams being plainly derived from the diagrams of the 6 × 6 and 10 × 10 squares, no explanation of them will be required, and it is

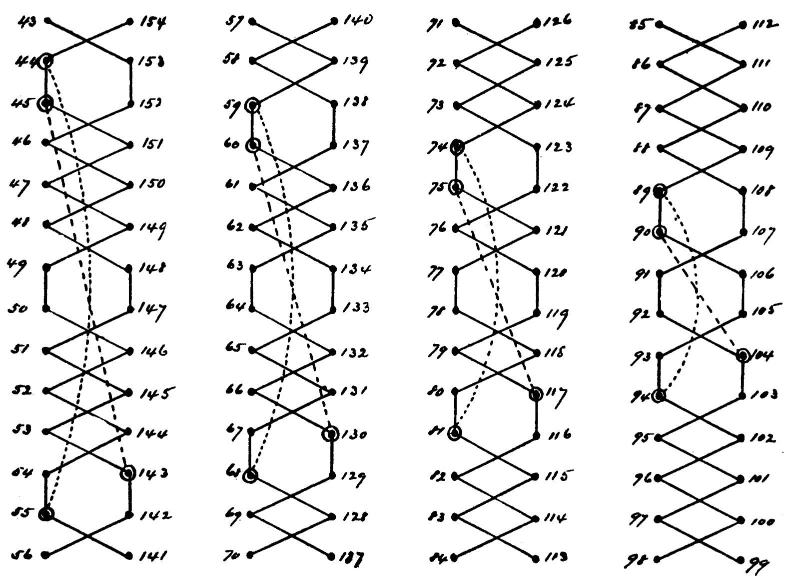

Fig. 65 (Second Part).

evident that the diagrammatic method may be readily applied to the construction of all sizes of even magic squares.

[TO BE CONCLUDED.]

W. S. ANDREWS.

NEW YORK.